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CURVATURE TENSORS IN KAEHLER MANIFOLDS(1)

BY

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ABSTRACT. Curvature tensors of Kaehler type (or type K) are defined on a hermitian vector space and it has been proved that the real vector space $\mathfrak{L}_K(V)$ of curvature tensors of type K on V is isomorphic with the vector space of symmetric endomorphisms of the symmetric product of V^+, where $V^{\mathbf{C}} = V^+ \oplus V^-$ (Theorem 3.6). Then it is shown that $\mathfrak{L}_K(V)$ admits a natural orthogonal decomposition (Theorem 5.1) and hence every $L \in \mathfrak{L}_K(V)$ is expressed as $L = L_1 + L_W + L_2$. These components are explicitly determined and then it is observed that L_W is a certain formal tensor introduced by Bochner. We call L_W the *Bochner-Weyl* part of L and the space of all these L_W is called the *Weyl subspace* of $\mathfrak{L}_K(V)$.

Introduction. Singer and Thorpe [6] established a natural decomposition of curvature tensors on an n-dimensional vector space with an inner product and then studied the curvature tensor of a 4-dimensional Einstein space and its canonical form. Nomizu [5] using this decomposition discussed generalised curvature tensor fields satisfying the second Bianchi identity (which are called *proper*) on a Riemannian manifold.

The purpose of this paper is to give a similar decomposition of curvature tensors of Kaehler type on a $2n$-dimensional hermitian vector space and then study curvature tensors on a Kaehler manifold. In this paper the study is purely local in nature. The contents are as follows:

In §1 some basic facts about hermitian vector spaces $(V, J, \langle\ ,\ \rangle)$ are given. Kaehler metric on a complex manifold is defined and the properties of its curvature tensor are stated in §2. In §3 a curvature tensor of type K on a hermitian vector space $(V, J, \langle\ ,\ \rangle)$ is defined and then proved that the real vector space $\mathfrak{L}_K(V)$ of

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all curvature tensors of type K on V is isomorphic to the real vector space $s(V^+ \cdot V^+)$ of all symmetric endomorphisms of the symmetric product of V^+ (Theorem 3.6). §4 deals with Ricci curvature $K(L)$ and holomorphic sectional curvature k_L of L in $\mathfrak{L}_K(V)$. In §5 we have proved that $\mathfrak{L}_K(V)$ is an orthogonal direct sum of three subspaces (Theorem 5.1) and hence every $L \in \mathfrak{L}_K(V)$ can be uniquely written as $L = L_1 + L_W + L_2$, these components being determined explicitly (Theorem 5.4); L_W is observed to be a certain formal tensor introduced by Bochner [7] and hence we call L_W the Bochner-Weyl part of L. The subspace of these L_W's is denoted by $\mathfrak{L}_K^W(V)$ and called the Weyl subspace. In a forthcoming paper we study curvature tensors in $\mathfrak{L}_K^W(V)$.

1. **Algebraic preliminaries.** Let V be a $2n$-dimensional real vector space with a complex structure J. Let $\langle\, ,\rangle$ be an inner product of hermitian type on V. Denote by $V^{\mathbf{C}}$ the complexification of V and $^-$ denotes the conjugation in $V^{\mathbf{C}}$ with respect to (for short, w.r.t.) V. Further, let L be a real endomorphism of V. Then we can extend L to a complex linear map of $V^{\mathbf{C}}$ to itself; in particular, J extends to $V^{\mathbf{C}}$. Since $J^2 = -$ identity, J is a nonsingular semisimple endomorphism of $V^{\mathbf{C}}$ with eigenvalues $+i$ and $-i$, where $i = \sqrt{-1}$ (i is $\sqrt{-1}$ always unless i is a subscript or superscript). Hence $V^{\mathbf{C}} = V^+ \oplus V^-$ where

$$V^+ = \{u \in V^{\mathbf{C}} \mid Ju = iu\} \quad \text{and} \quad V^- = \{u \in V^{\mathbf{C}} \mid Ju = -iu\}.$$

Moreover, $\overline{V^{+}} = V^-$ and hence complex dimension of $V^+ = n =$ complex dimension of V^-.

The hermitian inner product $\langle\, ,\rangle$ on V can be extended to $V^{\mathbf{C}}$ as a symmetric complex bilinear form (also denoted by $\langle\, ,\rangle$) and satisfies $\langle V^\pm, V^\pm\rangle = 0$. If we define $(u, v) = \langle u, \bar{v}\rangle$ for $u, v \in V^+$, then $(\, ,\,)$ is a positive definite hermitian form on V^+.

2. **Curvature tensor of a Kaehler metric.** Let M be a differentiable (i.e. c^∞-) manifold of dimension n. $T_p(M)$ denotes the tangent space of M at p which is an n-dimensional vector space. If M is a complex manifold of complex dimension n then the tangent space $T_Z(M)$ at Z of M as a c^∞-manifold is a real vector space of real dimension $2n$ with a complex structure J. A Riemannian metric g on M is called a *hermitian metric* if the inner product g_Z on $T_Z(M)$ is of hermitian type for each $Z \in M$.

Let g be a hermitian metric on M. Define $\Omega(u,v) = g(Ju, v)$ for $u, v \in T_Z(M)$. Then Ω is a real differential 2-form on M of type (1, 1) and is of maximal rank. We call Ω the fundamental 2-form of the hermitian manifold (M, g). A hermitian manifold (M, g) is *Kaehlerian* if the fundamental 2-form Ω is closed, i.e. $d\Omega = 0$.

We give two important examples of Kaehler manifolds.

(a) The complex number space $\mathbf{C}^n$ with the usual inner product is a Kaehler manifold and $\Omega = \sqrt{-1}\ \Sigma dZ^j \wedge \overline{dZ^j}$ where (Z^i) are the canonical coordinates of $\mathbf{C}^n$.

(b) The complex projective space $\mathbf{P}^n(\mathbf{C})$ with the Fubini-Study metric is a Kaehler manifold and so is every complex submanifold of $\mathbf{P}^n(\mathbf{C})$ (cf. [1, p. 170]).

Let ∇ denote the connection defined by the metric g. Then the curvature tensor R of ∇ is a $(1, 3)$ tensor. That is, $R(X, Y, Z)$ is a vector field. If we write $R(X, Y, *) = R(X, Y)^*$ then $R(X, Y)$, for fixed X, Y, is a $(1, 1)$ tensor. Moreover for fixed X, Y, $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ is a $(1, 1)$ tensor (cf. [2, p. 133]).

The following gives the properties of the curvature tensor R of the Kaehler metric g on a manifold M.

Proposition 2.1 ([3], [4]). *Let (M, g) be a Kaehler manifold and R be the curvature tensor of g. Then (1) $R(X, Y)J = JR(X, Y)$ and (2) $R(JX, JY) = R(X, Y)$ for any two vector fields X, Y on M.*

3. **Generalized curvature tensors.** In this section, we want to study curvature tensors having properties given in Proposition 2.1 in a more general algebraic setup.

Let V be a $2n$-dimensional real vector space with an inner product $\langle\, ,\rangle$. Denote by $o(V)$ the vector space of skew symmetric endomorphisms of V. We define a curvature tensor L on V as an $o(V)$-valued 2-form on V. Then for each $x, y \in V$, $L(x, y) \in \mathrm{End}(V)$ such that

(1) $L(x, y)$ is bilinear in x and y;

(2) $L(x, y) = -L(y, x)$;

(3) $\langle L(x, y)z, w\rangle + \langle z, L(x, y)w\rangle = 0$.

Now assume V admits a complex structure J and a hermitian inner product $\langle\, ,\rangle$.

A curvature tensor L on V is said to be of *type* K on V if L satisfies the following:

(1) $L(Jx, Jy) = L(x, y)$ for all $x, y \in V$;

(2) $L(x, y)J = JL(x, y)$;

(3) $L(x, y)z + L(y, z)x + L(z, x)y = 0$ for all $x, y, z \in V$ (Bianchi identity).

Denote by $\mathfrak{L}_K(V)$ the real vector space of all curvature tensors of type K on V. Our goal is to understand $\mathfrak{L}_K(V)$.

Let $L \in \mathfrak{L}_K(V)$. Extend L to $V^{\mathbf{C}}$. Then L is a skew symmetric bilinear map on $V^{\mathbf{C}}$ with values in $\mathrm{End}(V^{\mathbf{C}})$. Extend J and $\langle\, ,\rangle$ to $V^{\mathbf{C}}$ as in §1. Then

(1) $L(Jx, Jy) = L(x, y)$ for all $x, y \in V$ is equivalent to $L(V^{\pm}, V^{\pm}) = 0$.

(2) $L(x, y)J = JL(x, y)$ implies $L(x, y)$ leaves $V^{\pm}$ invariant.

(3) Bianchi identity holds in the complex case.

Since $L(V^{\pm}, V^{\pm}) = 0$, the Bianchi identity gives

(a) $L(x, \bar{y})\bar{z} = L(x, \bar{z})\bar{y}$ and hence $L(x, \bar{y})\bar{z}$ is symmetric in y and z, and

(a′) $L(x, \bar{w})y = L(y, \bar{w})x$ and hence $L(x, \bar{w})y$ is symmetric in x and y.

Note that since (a′) is obtained from (a) by conjugation a curvature tensor L satisfies Bianchi identity if and only if L satisfies (a). Now define the map F_L: $V^+ \times V^+ \times V^+ \times V^+ \to \mathbf{C}$ by

$$(3.1) \qquad F_L(y, z, x, w) = L(y, \bar{z}, x, \bar{w}) = \langle L(x, \bar{w})\bar{z}, y\rangle \quad \text{for all } x, y, z, w \in V^+.$$

Then clearly this map has the properties

(a) F_L is bilinear in x and y and anti-bilinear in w and z;

(b) F_L is symmetric in x and y and also in z and w.

Since given any $F: V^+ \times V^+ \times V^+ \times V^+ \to \mathbf{C}$ satisfying (a) and (b) there exists a unique $L \in \mathfrak{L}_K(V)$ such that $F = F_L$ (as given in (3.1)); we call such an F also a curvature tensor of type K on V. Note that $F_L(y, z, x, w) = F_L(x, w, y, z)$ (from (b)).

Define hermitian inner product in $V^{\pm}$ by $(x, y) = \langle x, \bar{y}\rangle$. Next we extend $(\, , \,)$ on V^+ to $V^+ \otimes V^+$ by defining $(x \otimes y, w \otimes z) = (x, w)(y, z)$.

Now we have

Proposition 3.2. *If $f(w, z)$ is an anti-bilinear function on V^+ then there exists $u \in V^+ \otimes V^+$ such that $f(w, z) = (u, w \otimes z)$.*

Proof. Obvious.

Since, for every fixed $x, y \in V^+$, $F_L(y, *, x, *)$ is anti-bilinear on V^+, by Proposition 3.2, there exists $u(x, y) \in V^+ \otimes V^+$ such that $F_L(y, z, x, w) = (u(x, y), w \otimes z)$. Since F_L is bilinear and symmetric in x and y the map u: $V^+ \times V^+ \to V^+ \otimes V^+$ sending (x, y) to $u(x, y)$ is bilinear and symmetric. Hence there exists a linear endomorphism U': $V^+ \otimes V^+ \to V^+ \otimes V^+$ such that $U'(x \otimes y) = u(x, y)$, and hence we have

$$(3.3) \qquad (U'(x \otimes y), w \otimes z) = F_L(y, z, x, w).$$

Moreover, using (3.3) and (3.1) we see easily that U' is a symmetric endomorphism of $V^+ \otimes V^+$ w.r.t. $(\, , \,)$. Now we carry this U' one more step forward.

Consider the symmetric product $V^+ \cdot V^+$ of V^+. $V^+ \cdot V^+$ is, by definition, the quotient of $V^+ \otimes V^+$ by the subspace A consisting of all skew symmetric tensors. Denote by π the canonical projection of $V^+ \otimes V^+$ onto $V^+ \cdot V^+$. Then $t \in V^+ \otimes V^+$ implies $t = s + a$ where s is a symmetric tensor and $a \in A$. Define an inner product in $V^+ \cdot V^+$ by $(\pi(t), \pi(t')) = (s, s')$ where s and s' are respectively the symmetric parts of t and t' and (s, s') is the inner product of s

and s' in $V^+ \otimes V^+$. Since symmetric and skew symmetric tensors are perpendicular, the inner product in $V^+ \cdot V^+$ satisfies the condition $(t, t') = (\pi(t), \pi(t')) + (a, a')$. Moreover, from our definition it follows that $(x \cdot y, w \cdot z) = \frac{1}{2}\{(x, w)(y, z) + (x, z)(y, w)\}$ where we have put $x \cdot y = \pi(x \otimes y)$ for all $x, y \in V^+$.

Now we claim that there exists a symmetric linear map U: $V^+ \cdot V^+ \to V^+ \cdot V^+$ such that $F_L(y, z, x, w) = (U(x \cdot y), w \cdot z)$. In fact, since U': $V^+ \otimes V^+ \to V^+ \otimes V^+$ is defined by $U'(x \otimes y) = u(x, y)$ and, since $u(x, y)$ is symmetric in x and y, U' maps kernel π into zero and hence U' induces a unique linear map U: $V^+ \cdot V^+ \to V^+ \cdot V^+$ such that $U(x, y) = \pi(U'(x \otimes y))$. It remains to show

$$(3.4) \qquad (U(x \cdot y), w \cdot z) = (U'(x \otimes y), w \otimes z).$$

But we have $(U'(x \otimes y), w \otimes z) = (U(x \cdot y), w \cdot z) + (a, a')$ where a and a' denote the skew symmetric parts of $U'(x \otimes y)$ and $w \otimes z$ respectively. Since $F_L(y, v, x, u) = (U'(x \otimes y), u \otimes v)$ and since F_L is symmetric in u and v, $U'(x \otimes y)$ is orthogonal to the subspace A of skew symmetric tensors and hence $U'(x \otimes y)$ is a symmetric tensor and hence $a = 0$. This completes the proof of our claim. Moreover, we can prove that this U is a symmetric endomorphism of $V^+ \cdot V^+$.

Denote by $s(V^+ \cdot V^+)$ the vector space of all symmetric endomorphisms of $V^+ \cdot V^+$. In summary, starting with a curvature tensor $L \in \mathfrak{L}_K(V)$ we obtained a $U_L \in s(V^+ \cdot V^+)$ such that

$$(3.5) \qquad (U_L(x \cdot y), w \cdot z) = F_L(y, z, x, w) = \langle L(y, \bar{z})\bar{w}, x\rangle .$$

Denote this map from $\mathfrak{L}_K(V)$ into $s(V^+ \cdot V^+)$ by ϕ. Now we have the following:

Theorem 3.6. *The map ϕ defined above (by (3.5)) is an isomorphism of $\mathfrak{L}_K(V)$ onto $s(V^+ \cdot V^+)$.*

Before proceeding with the proof of this theorem we give some facts about bases of V^+ and $V^+ \cdot V^+$.

Let $e_1, e_2, \cdots, e_n$ be an orthonormal basis of V^+, an n-dimensional hermitian vector space, w.r.t. $(\, , \,)$. Then $\{e_1, e_2, \cdots, e_n, \bar{e}_1, \bar{e}_2, \cdots, \bar{e}_n\}$ forms a basis of $V^{\mathbf{C}}$ and we have

(1) $\{e_i \cdot e_j, \ i \leq j, \ i, j = 1, 2, \cdots, n\}$ is a basis of $V^+ \cdot V^+$.

(2) $\{e_i \cdot e_i, \sqrt{2}\, e_i \cdot e_j, \ i < j, \ j = 1, 2, \cdots, n\}$ is an orthonormal basis of $V^+ \cdot V^+$ w.r.t. $(\, , \,)$.

(3) Putting $f_i = (e_i + \bar{e}_i)/\sqrt{2}$, $Jf_i = \sqrt{-1}\,(e_i - \bar{e}_i)/\sqrt{2}$, $i = 1, 2, \cdots, n$, $\{f_i, Jf_i\}$ is a basis for V as a $2n$-dimensional real vector space.

(4) $\overline{L(\bar{e}_i, e_j)} = L(e_i, \bar{e}_j)$.

Proof of Theorem. That ϕ is an injective real linear map is clear from (3.5). We have to show ϕ is onto. That is, given $U \in s(V^+ \cdot V^+)$, we have to define for each $x, y \in V$ a real endomorphism $L(x, y)$ of V such that $L \in \mathfrak{L}_K(V)$ and L when extended to a complex linear map of V^+ should satisfy (3.5).

Taking $x = e_k$, $y = e_l$ $(k \leq l)$, and $w = e_m$, $z = e_p$ in (3.5) we get

$$(3.7) \qquad L_{l\bar{p}k\bar{m}} = \begin{cases} U_{mm,kl} & \text{if } p = m, \\ U_{mp,kl} & \text{if } p \neq m \ (k \leq l,\ m \leq p). \end{cases}$$

Because of symmetry in k and l and in p and m (3.7) defines L for all suffixes.

Now we want to define the corresponding real endomorphism L of V so that $L \in \mathfrak{L}_K(V)$. Let $\{f_i, Jf_i, i = 1, 2, \cdots, n\}$ be real basis of V such that $f_k = e_k + \bar{e}_k$ and $Jf_k = i(e_k - \bar{e}_k)$. Then we compute L in terms of components of U (by (3.7)) as in

(1) $\langle L(f_k, f_m)f_p, f_l\rangle = L_{l\bar{p}k\bar{m}} - L_{p\bar{l}k\bar{m}} + L_{p\bar{l}m\bar{k}} - L_{l\bar{p}m\bar{k}}$.

Using (3.7) and the fact that U is hermitian we can show the right-hand side of (1) is real.

(2) $\langle L(f_k, Jf_m)f_p, f_l\rangle = i\{L_{p\bar{l}m\bar{k}} - L_{l\bar{p}k\bar{m}} + L_{p\bar{l}k\bar{m}} - L_{l\bar{p}m\bar{k}}\}$ which is also real.

(3) $\langle L(f_k, f_m)Jf_p, f_l\rangle = i\{L_{p\bar{l}k\bar{m}} - L_{l\bar{p}k\bar{m}} + L_{p\bar{l}m\bar{k}} - L_{l\bar{p}m\bar{k}}\}$. This is also real from (2).

(4) $\langle L(f_k, Jf_m)Jf_p, f_l\rangle = -\{L_{p\bar{l}k\bar{m}} + L_{l\bar{p}m\bar{k}} - L_{l\bar{p}k\bar{m}} - L_{p\bar{l}m\bar{k}}\}$. This is also real.

From (2) and (3) we get

(5) $\langle L(f_k, Jf_p)f_m, f_l\rangle = \langle L(f_k, f_m)Jf_p, f_l\rangle$.

Now using these relations (1)–(5) we define L on the basis elements $\{f_k, Jf_k\}$ of V as follows: Define $L(f_k, f_m)$ by (1) and (3); $L(f_k, Jf_m)$ by (2) and (4); $L(Jf_k, Jf_m) = L(f_k, f_m)$ and $L(f_k, f_m)Jf_p = JL(f_k, f_m)f_p$. Then this L is defined as a real endomorphism of V in terms of the components of given U and L has the properties

(a) $L(f_i', f_j') = -L(f_j', f_i')$,

(b) $L(Jf_i, Jf_j) = L(f_i, f_j)$, and

(c) $L(f_i', f_j')J = JL(f_i', f_j')$, where $\{f_1', f_2', \cdots, f_n', f_{n+1}', \cdots, f_{2n}'\}$ denotes $\{f_i, Jf_i\}$, $i = 1, 2, \cdots, n$.

Hence L belongs to $\mathfrak{L}_k(V)$ provided L satisfies the Bianchi identity, i.e.

$$(3.8) \qquad L(x, y)z + L(y, z)x + L(z, x)y = 0.$$

We verify this on basis elements of V. Let Σ_L denote the left-hand side of (3.8).

Case (a). Let $x = f_k$, $y = f_m$, $z = f_p$ and $w = f_l$. Then from relations (1)–(5) we get

$$\begin{aligned}\langle \Sigma_L, f_l\rangle = {} & L_{l\bar{p}k\bar{m}} + L_{p\bar{l}m\bar{k}} - L_{p\bar{l}k\bar{m}} - L_{l\bar{p}m\bar{k}} \\ & + L_{l\bar{k}m\bar{p}} + L_{k\bar{l}p\bar{m}} - L_{k\bar{l}m\bar{p}} - L_{l\bar{k}p\bar{m}} \\ & + L_{l\bar{m}p\bar{k}} + L_{m\bar{l}k\bar{p}} - L_{m\bar{l}p\bar{k}} - L_{l\bar{m}k\bar{p}}\end{aligned}$$

which can be seen to be zero using (3.7) and U is hermitian.

Similarly in the other cases

(b) $x = f_k$, $y = f_m$, $z = f_p$ and $w = Jf_l$,
(c) $x = f_k$, $y = Jf_m$, $z = f_p$ and $w = f_l$ or Jf_l,
(d) $x = f_k$, $y = Jf_m$, $z = Jf_k$;
(e) $x = Jf_k$, $y = Jf_m$, $z = Jf_p$,

etc., we can show that $\langle \Sigma_L, w\rangle = 0$ using $L(Jx, Jy) = L(x, y)$ and $L(x, y)J = JL(x, y)$.

Thus L as defined above belongs to $\mathfrak{L}_K(V)$ and in view of (3.5) we have $\phi(L) = U$, hence ϕ is surjective.

This completes the proof of the theorem.

Remarks. (1) Because of this isomorphism ϕ between $\mathfrak{L}_K(V)$ and $s(V^+ \cdot V^+)$ we call elements of $s(V^+ \cdot V^+)$ also curvature tensors of type K and we identify $\mathfrak{L}_K(V)$ and $s(V^+ \cdot V^+)$. Sometimes for $L \in \mathfrak{L}_K(V)$ we denote its image under ϕ simply by L' instead of U_L or $\phi(L)$.

(2) If $\mathfrak{L}_R(V)$ denotes the vector space of curvature tensors on V satisfying the Bianchi identity (here V is a real inner product space) then $\mathfrak{L}_R(V)$ is isomorphic with a *proper* subspace of $s(V \wedge V)$ (cf. [6]). But in the Kaehler case we have $\mathfrak{L}_K(V)$ is isomorphic with $s(V^+ \cdot V^+)$.

4. **Ricci and holomorphic sectional curvatures of L.** We define a linear map $K: \mathfrak{L}_K(V) \to \mathrm{End}(V^+)$, called the Ricci map, as follows.

For every $L \in \mathfrak{L}_K(V)$, $K(L)$ is an endomorphism of V^+ such that $K(L)x = \sum_{s=1}^n L(x, \bar{e}_s)e_s$ where $x \in V^+$ and $\{e_i\}$ $(i = 1, 2, \cdots, n)$ is an orthonormal basis of V^+ w.r.t. (,). We call $K(L)$ the Ricci tensor of L. It is easily seen that $K(L)$ is a symmetric endomorphism of V^+ w.r.t. (,). If we define $K(x, y) = (K(L)x, y)$ then $K(x, y)$ is a symmetric bilinear form on V^+, called the *Ricci form* of L in $\mathfrak{L}_K(V)$.

Let $\mathbf{P}(V^+)$ denote the complex projective space of complex lines in V^+. For any $L \in \mathfrak{L}_K(V)$ we define a function $k_L: \mathbf{P}(V^+) \to \mathbf{C}$ as follows.

Let v be any vector in a complex line P of V^+ such that $\|v\| = 1$ w.r.t. (,). Then

$$k_L(P) = \langle L(\bar{v}, v)\bar{v}, v \rangle = -(L'(v \cdot v), v \cdot v).$$

Then $k_L(P)$ is well defined for any P in $\mathbf{P}(V^+)$.

Remark. $k_L(P)$ is, in fact, a real number because L' is hermitian. Moreover, if we write $v \in V^+$ as $v = (X - iJX)/\sqrt{2}$, $X, JX \in V$ (the corresponding real 2-plane in V is spanned by $X/\sqrt{2}$, $JX/\sqrt{2}$) then $\langle L(\bar{v}, v)\bar{v}, v\rangle = \langle L(X, JX)JX, X\rangle \in \mathbf{R}$. So our definition of k_L is the usual definition of holomorphic sectional curvature function.

Definition of symmetric (wedge) product of two endomorphisms. Let V be a vector space and $A, B \in \mathrm{End}(V)$. Let $V \cdot V = V \otimes V/A'$ (resp. $V \wedge V = V \otimes V/N$) where A' is the subspace of $V \otimes V$ generated by $x \otimes y - \;\; \otimes x$ for all x, $y \in V$ (resp. N is the subspace of $V \otimes V$ generated by $x \otimes x$ for all $x \in V$). Then $A \otimes B \in \mathrm{End}(V \otimes V)$ is defined by $(A \otimes B)(x \otimes y) = Ax \otimes By$ and $A \otimes B + B \otimes A$ leaves A' (resp. N) invariant and hence induces an endomorphism of $V \cdot V$ (resp. $V \wedge V$). We denote this endomorphism by $A \cdot B$ (resp. $A \wedge B$). Then we have $(A \cdot B)(x \cdot y) = (Ax \cdot By) + (Ay \cdot Bx)$.

Now let $A, B \in s(V^+)$. Then we can show that $A \cdot B \in s(V^+ \cdot V^+)$. In fact

$$((A \cdot B)(x \cdot y), w \cdot z) = (Ax \cdot By, w \cdot z) + (Ay \cdot Bx, w \cdot z)$$

$$= (x \cdot y, Aw \cdot Bz) + (y \cdot x, Aw \cdot Bz) = (x \cdot y, (A \cdot B)(w \cdot z)).$$

Therefore, by Theorem 3.6, $A \cdot B$ corresponds to a curvature tensor of type K on V and we denote this by $L_{A \cdot B} \in \mathfrak{L}_K(V)$.

Proposition 4.1. *Let $L_1, L_2 \in \mathfrak{L}_K(V)$ such that*

$$\langle L_1(\bar{v}, v)\bar{v}, v\rangle = \langle L_2(\bar{v}, v)\bar{v}, v\rangle \quad \textit{for all } v \in V^+.$$

Then $L_1 = L_2$.

This is the same as Proposition 7.1 in [3, p. 166].

Proposition 4.2. *Let $L \in \mathfrak{L}_K(V)$. Then k_L is constant (say c) if and only if $L = -cL_{A \cdot B}$ with $A = B = I/\sqrt{2} \in s(V^+)$, where I denotes the identity endomorphism of V^+.*

Proof. $(\Rightarrow)$ Let $P \in \mathbf{P}(V^+)$. Then $k_L(P) = -(L'(v \cdot v), v \cdot v) = c$.

On the other hand, $I \cdot I/2 \in s(V^+ \cdot V^+)$ and

$$\langle L_{I \cdot I/2}(\bar{v}, v)\bar{v}, v\rangle = -\langle (I \cdot I/2)(v \cdot v), v \cdot v\rangle = -1 \quad \text{for all } v. \tag{4.3}$$

Hence $\langle L(\bar{v}, v)\bar{v}, v\rangle = c = -c\langle L_{I \cdot I/2}(\bar{v}, v)\bar{v}, v\rangle$ for all $v \in V^+$ and hence $L = -cL_{I \cdot I/2}$.

Conversely, given $L = -cL_{I.I/2}$ we can see easily from (4.3) that $k_L(P) = c$ for all $P \in \mathbf{P}(V^+)$.

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis of V^+ w.r.t. (,) $K_{A \cdot B}$ denotes the Ricci tensor of $L_{A \cdot B}$. Then

$$K_{A \cdot B} e_j = \sum_{s=1}^{n} L_{A \cdot B}(e_j, \bar{e}_s)e_s, \qquad K_{A \cdot B} \in s(V^+).$$

Then

$$(K_{A \cdot B})_{ij} = (K_{A \cdot B}e_j, e_i) = -\sum_{s=1}^{n} ((A \cdot B)(e_j \cdot e_s), e_s \cdot e_i)$$

$$= -\tfrac{1}{2}\{A_{ij} \operatorname{Tr} B + B_{ij} \operatorname{Tr} A + (AB)_{ij} + (BA)_{ij}\}.$$

So we get

(4.4) $$-2K_{A \cdot B} = (\operatorname{Tr} A)B + (\operatorname{Tr} B)A + AB + BA$$

and hence

(4.5) $$-\operatorname{Tr} K_{A \cdot B} = \operatorname{Tr} A \operatorname{Tr} B + \operatorname{Tr} AB.$$

Recalling L' is the image of L under ϕ in Theorem 3.6, we can prove that

(4.6) $$\operatorname{Tr} K(L) = 2 \operatorname{Tr} L' = 2 \operatorname{Tr} U_L.$$

We have inner product in $\mathfrak{L}_K(V)$ defined from that of V^+ as

$$(L_1, L_2) = \sum_{i,j,k,l} (L_1)_{ijkl}(L_2)_{ijkl}.$$

The inner product (,) of V^+ also induces an inner product (,) on $s(V^+ \cdot V^+)$ (observe that, in general, if W is a complex vector space with an inner product and $s(W)$ is the real vector space of all symmetric endomorphisms of W then we can define an inner product (,) in $s(W)$ by $(S, T) = \operatorname{Tr}(ST)$). Then we have

(4.7) $$(L_1, L_2) = \sum_{i,j,k,l} (L_1)_{ijkl}(L_2)_{ijkl} = \operatorname{Tr}(L_1', L_2').$$

Let $\mathfrak{L}_K^1(V) = \{L \in \mathfrak{L}_K(V) |\ L' = \lambda I \cdot I,\ \lambda \in \mathbf{R}\}$. Then $\mathfrak{L}_K^1(V)$ is a real 1-dimensional subspace of $\mathfrak{L}_K(V)$. Let $\operatorname{Orth}(\mathfrak{L}_K^1(V))$ denote the orthogonal complement of $\mathfrak{L}_K^1(V)$ in $\mathfrak{L}_K(V)$ w.r.t. inner product defined above. Then in view of (4.6) and (4.7) we have

$$\operatorname{Orth}(\mathfrak{L}_K^1(V)) = \{L \in \mathfrak{L}_K(V) \mid \operatorname{Tr} K(L) = 0\}.$$

Hence we have the decomposition

$$\mathfrak{L}_K(V) = \mathfrak{L}_K^1(V) \oplus \text{Orth}\,(\mathfrak{L}_K^1(V)).$$

5. **Decomposition of the space $\mathfrak{L}_K(V)$.** Recall $\mathfrak{L}_K^1(V) = \{L \in \mathfrak{L}_K(V) |\ L' = \lambda I \cdot I,\ \lambda \in \mathbf{R}\}$. Let $\mathfrak{L}_K^W(V) = \{L \in \mathfrak{L}_K(V) |\ K(L) = 0\}$ where K is the Ricci map. Then clearly $\mathfrak{L}_K^W(V) \subset \text{Orth}\,(\mathfrak{L}_K^1(V))$. Now let $\mathfrak{L}_K^2(V)$ be the orthogonal complement of $\mathfrak{L}_K^W(V)$ in $\text{Orth}\,(\mathfrak{L}_K^1(V))$. Therefore we have the natural decomposition of $\mathfrak{L}_K(V)$ as $\mathfrak{L}_K(V) = \mathfrak{L}_K^1(V) \oplus \mathfrak{L}_K^W(V) \oplus \mathfrak{L}_K^2(V)$. Now we have the following:

Theorem 5.1. *Let V be a real $2n$-dimensional hermitian vector space. Let $\mathfrak{L}_K(V)$ be the real vector space of all curvature tensors of type K on V. Then $\mathfrak{L}_K(V)$ has the decomposition*

$$\mathfrak{L}_K(V) = \mathfrak{L}_K^1(V) \oplus \mathfrak{L}_K^W(V) \oplus \mathfrak{L}_K^2(V) \quad (\textit{orthogonal})$$

where

(1) $\mathfrak{L}_K^1(V) = \{L \in \mathfrak{L}_K(V) |\ L$ *has constant holomorphic sectional curvature*$\}$;
(2) $\mathfrak{L}_K^W(V) = \{L \in \mathfrak{L}_K(V) |\ L$ *has Ricci curvature zero*$\}$;
(3) $\mathfrak{L}_K^W(V) \oplus \mathfrak{L}_K^2(V) = \{L \in \mathfrak{L}_K(V) |\ \text{Tr}\, K(L) = 0\}$;
(4) $\mathfrak{L}_K^1(V) \oplus \mathfrak{L}_K^W(V) = \{L \in \mathfrak{L}_K(V) |\ K(L) = \lambda I,\ \lambda \in \mathbf{R}\}$.

Proof. All except (4) are proved above.

Suppose $L \in \mathfrak{L}_K^1(V) \oplus \mathfrak{L}_K^W(V)$. Then $L = L_1 + L_W$ and hence $K(L) = K(L_1)$ because K is linear. But since $L_1' = I \cdot I$ and using (4.4) we get $K(L_1) = -(n+1)I = K(L)$. Conversely, let $K(L) = \lambda I$, $\lambda \in \mathbf{R}$. Let L_1 be the curvature tensor corresponding to $L_1' = \mu I \cdot I \in s(V^+ \cdot V^+)$ with $\mu = -\lambda/(n+1)$. Then $L_1 \in \mathfrak{L}_K^1(V)$ and $K(L - L_1) = 0$. Hence $L - L_1 \in \mathfrak{L}_K^W(V)$. So $L = L_1 + L_W$.

Recall that a Kaehler metric g on a manifold M is called Einstein if R, the Ricci tensor of g, is a scalar multiple of g at each point p of M.

Corollary 5.2. *Suppose (M, g) is a Kaehler-Einstein manifold. Then the curvature tensor L (of g) belongs to*

$$\mathfrak{L}_K^1(T_p(M)) \oplus \mathfrak{L}_K^W(T_p(M)), \quad \forall p \in M \quad (\forall\ \textit{means "for every"})$$

This is immediate from (4) of Theorem 5.1.

Corollary 5.3. *If M is a connected Kaehler-Einstein manifold of complex dimension ≥ 2 (i.e. $K(L)_p = \lambda(p) I$ for every $p \in M$) then λ is a constant function.*

Proof. By (5.2), $L_p \in \mathfrak{L}_K^1(T_p(M)) \oplus \mathfrak{L}_K^W(T_p(M))$ and hence $R_p = K(L)_p = \lambda(p) I$, i.e. $R_{i\bar{j}} = \lambda(p) g_{i\bar{j}}$ (in local coordinates). Then the first Chern class of M is represented by $\psi = -(1/2\pi)\lambda(p)\Omega$ where $\Omega = i g_{i\bar{j}}\, d\bar{z}^i \wedge dz^j$ is the fundamental 2-form of g and $\psi = (1/2\pi i) R_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$ is the Ricci form of g. Since ψ is a closed 2-form we have $0 = d\psi = -(1/2\pi)\, d\lambda \wedge \Omega$. Then $d\lambda = 0$ because wedge product with Ω is an isomorphism of complex vector spaces for complex dimension of $M \geq 2$. Hence λ is a constant function on M.

Now we give the components of $L \in \mathfrak{L}_K(V)$ in the decomposition.

Theorem 5.4. *Let $L \in \mathfrak{L}_K(V)$ and $K(L)$ be the Ricci tensor of L. Further let $\lambda = \operatorname{Tr} K(L)$ and L_1, L_W, L_2 be the components of L in the decomposition of Theorem 5.1. Then*

(5.5) $$L_1' = -\lambda(I \cdot I)/n(n+1),$$

(5.6) $$L_2' = -2(K \cdot I)/(n+2) + 2\lambda(I \cdot I)/n(n+2),$$

(5.7) $$L_W' = L' + 2(K \cdot I)/(n+2) - \lambda(I \cdot I)/(n+1)(n+2).$$

(Here $K = K(L)$ and I = identity endomorphism of V^+.)

Proof. Let R_1, R_2, R_W denote respectively the right-hand sides of (5.5), (5.6) and (5.7). We show that $R_1 \in \mathfrak{L}_K^1(W)$, $R_W \in \mathfrak{L}_K^W(V)$ and $R_2 \in \mathfrak{L}_K^2(V)$. Clearly $R_1 \in \mathfrak{L}_K^1(V)$.

$$K(R_W) = K(L) + 2K(L_{K \cdot I})/(n+2) - \lambda K(L_{I \cdot I})/(n+1)(n+2)$$

which is seen to be zero using (4.4). Hence $R_W \in \mathfrak{L}_K^W(V)$.

It remains to show $R_2 \in \mathfrak{L}_K^2(V)$.

First we observe that $\operatorname{Tr} K(R_2) = 0$ (use (4.5)) hence $R_2 \in \mathfrak{L}_K^W(V) \oplus \mathfrak{L}_K^2(V)$. It suffices to show that R_2 is perpendicular to $\mathfrak{L}_K^W(V)$. That is, $\operatorname{Tr}(R_2'R') = 0$ for all $R \in \mathfrak{L}_K^W(V)$.

We can choose an orthonormal basis $(e_1, \cdots, e_n)$ of V^+ such that $Ke_i = \lambda_i e_i$, $i = 1, 2, \cdots, n$ (since $K \in s(V^+)$). Then $(K \cdot I)(e_i \cdot e_j) = (\lambda_i + \lambda_j) e_i \cdot e_j$, $i, j = 1, 2, \cdots, n$, and $(I \cdot I)(e_i \cdot e_j) = 2e_i \cdot e_j$ and hence

(5.8) $$R_2(e_i \cdot e_j) = (-2/(n+2))\{(\lambda_i + \lambda_j) - 2\lambda/n\} e_i \cdot e_j \quad \forall i,j.$$

For $R \in \mathfrak{L}_K^W(V)$,

(5.9) $$\sum_s R_{sjsl} = 0 \quad \text{for all } l, j$$

(since $K(R) = 0$). In particular

(5.10) $$\sum_{s,j} R_{sjsj} = 0.$$

Then $(R_2, R) = \operatorname{Tr}(R_2'R') = \Sigma_{i,j,s,t}(R_2')_{ijst} R'_{stij}$ $(1 < j,\ s < t)$. Since $R'_{ijst} = 0$ unless $(i, j) = (s, t)$ we have

(5.11) $$\operatorname{Tr}(R_2'R') = \sum_{i,j} \{(\lambda_i + \lambda_j) - 2\lambda/n\} R'_{ijij} \qquad (i < j).$$

Now using (5.9), (5.10) and skew-symmetry in i and j, we can show that each term in (5.11) is separately zero, and hence $R_2 \in \mathfrak{L}_K^2(V)$. Since the decomposition is unique we have $R_1 = L_1'$, $R_2 = L_2'$, $R_W = L_W'$.

Remark. In the Kaehler case L_W' is given by

$$L_W' = L' + 2(K \cdot I)/(n+2) - (\mathrm{Tr}\, K)(I \cdot I)/(n+1)(n+2).$$

Then if we compute the components $(L_W')_{\alpha\beta\gamma\delta}$ w.r.t. an orthogonal coordinate (e_α) of V^+ we see that this is the formal tensor introduced by Bochner (which he denotes by K) up to sign (cf. [7, p. 161]). We call any $L_W \in \mathfrak{L}_K^W(V)$ a generalized *Boehner-Weyl tensor* and $\mathfrak{L}_K^W(V)$ is called the *Weyl subspace* of $\mathfrak{L}_K(V)$.

Now we can restate our Theorems 5.1 and 5.4 in terms of $s(V^+)$.

Theorem 5.12. *Let $(V, J, \langle\ ,\ \rangle)$ be a real $2n$-dimensional hermitian vector space. Let $\mathfrak{L}_K(V) = \mathfrak{L}_K^1(V) \oplus \mathfrak{L}_K^W(V) \oplus \mathfrak{L}_K^2(V)$ be the decomposition of Theorem 5.1. Then*

(a) $\mathfrak{L}_K^1(V) \oplus \mathfrak{L}_K^2(V) = \{L \in \mathfrak{L}_K(V) \mid L' = A \cdot I,\ A \in s(V^+)\}$.

(b) *The Ricci map $K: \mathfrak{L}_K(V) \to s(V^+)$ is onto,* $\ker K = \mathfrak{L}_K^W(V)$ *and K is bijective from $\mathfrak{L}_K^1(V) \oplus \mathfrak{L}_K^2(V)$ onto $s(V^+)$. Furthermore, $K(\mathfrak{L}_K^1(V)) = \{A \in s(V^+) \mid A = \lambda I,\ \lambda \in \mathbf{R}\}$ and $K(\mathfrak{L}_K^2(V)) = \{A \in s(V^+) \mid \mathrm{Tr}\, A = 0\}$.*

Proof of (a). Let $L = L_{A\cdot I} \in \mathfrak{L}_K(V)$ with $A \in s(V^+)$. Then $L' = A \cdot I$. Let $L = L_1 + L_2 + L_W$. To show $L_W = 0$ but finding $K = K(L')$ and $\lambda = \mathrm{Tr}\, K$ from $L' = A \cdot I$, and substituting for K and λ in L_W' we see easily L_W' is zero. Conversely, if $L = L_1 + L_2$ then $L_W' = 0$ and hence $L' = A \cdot I$ with $A = \lambda I/(n+1)(n+2) - 2K/(n+2) \in s(V^+)$.

Proof of (b). Let $B \in s(V^+)$. Write B uniquely as $B = B_0 + \mu I$ with $B_0 \in s(V^+)$ and $\mathrm{Tr}\, B_0 = 0$. Then, from (4.4), $K(L_{(-2B_0 \cdot I)/(n+2)}) = B_0$. Also we have $K(L_{-\zeta I\cdot I}) = \zeta(n+1)I$ for any real number ζ; in particular for $\zeta = -\mu/(n+1)$ we have $K(L_{-\mu/(n+1)I\cdot I} = \mu I$.

Set $A = -((2B_0 \cdot I)/(n+2) + (\mu I \cdot I)/(n+1))$. Then $K(L_A) = B_0 + \mu I = B$. Hence K is onto and the other statements are all obvious.

Corollary 5.13. *The real dimension of $\mathfrak{L}_K^2(V) = n^2 - 1$ where n = complex dimension of V.*

This is immediate from the bijectivity of $K: \mathfrak{L}_K^I(V) \oplus \mathfrak{L}_K^2(V) \to s(V^+)$.

Corollary 5.14. *Let V be a $2n$-dimensional real vector space with a complex structure J and an hermitian inner product $\langle\ ,\ \rangle$. Then the real dimension of $\mathfrak{L}_K^W(V) = n^2(n-1)(n+3)/4$ where $n = \dim \mathbf{C} V$.*

Proof. Since $\dim_{\mathbf{R}} s(V^+ \cdot V^+) = (n(n+1)/2)^2$, $\dim_{\mathbf{R}} \mathfrak{L}_K(V) = n^2(n+1)^2/4$. Hence by Corollary 5.13, $\dim_{\mathbf{R}} \mathfrak{L}_K^W(V) = n^2(n+1)^2/4 - n^2 = n^2(n-1)(n+3)/4$. Note that the Weyl subspace $\mathfrak{L}_K^W(V)$ is of real dimension 5 when V is of complex dimension 2. In general to give some characterisation of the Weyl subspace $\mathfrak{L}_K^W(V)$ seems quite difficult.

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